Kompleksna števila

1. Dana so kompleksna števila $z_1 = 1 + 2i$ in $z_2 = -2 + 5i$. Izračunaj in v kompleksni ravnini ravnini nariši kompleksna števila
   (a) $z_1 - z_2$
   (b) $z_1 z_2 + z_2$
2. Izračunaj.
   (a) $i + i^2 + i^3 + i^4 + i^5$
   (b) $2i \cdot 5i + 3i \cdot 2i^2 + 7i \cdot 8i^3 + 3i^2 \cdot 2i^3 + 2i^3 \cdot 4i^3$
   (c) $(\sqrt{3} + i\sqrt{2})(\sqrt{3} - i\sqrt{2}) + (1 - 3i)^2$

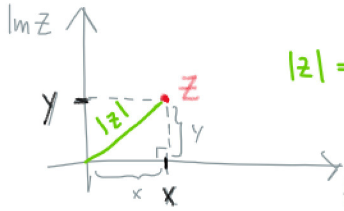
Definicija:

$z = x + yi$ [ali $a + bi$]

$x, y \in \mathbb{R}$

x... realni del [$\mathrm{Re}\, z$]

y... imaginarni del [$\mathrm{Im}\, z$]



$|z| = \sqrt{x^2 + y^2}$

i... imaginarna enota

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = \sqrt{-1} = i$

vzorec se ponavlja

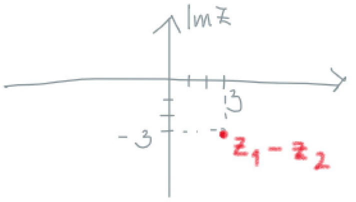
npr. $i^{19} = -i$

$i^{20} = 1$

1. a) $z_1 = 1 + 2i$ $\quad z_1 - z_2 = (1 + 2i) - (-2 + 5i)$

$z_2 = -2 + 5i$ $\quad = 1 + 2i + 2 - 5i$

$= \underline{\underline{3 - 3i}}$ $\quad x = 3$, $y = -3$



2. b) $2i \cdot 5i + 3i \cdot 2i^2 + 7i \cdot 8i^3 + 3i^2 \cdot 2i^3 + 2i^3 \cdot 4i^3 =$

$= 10i^2 + 6i^3 + 56i^4 + 6i^5 + 8i^6$

$= 10 \cdot (-1) + 6 \cdot (-i) + 56 \cdot 1 + 6 \cdot i + 8 \cdot (-1)$

$= -10 \cancel{-6i} + 56 \cancel{+6i} - 8$

$= \underline{\underline{38}}$

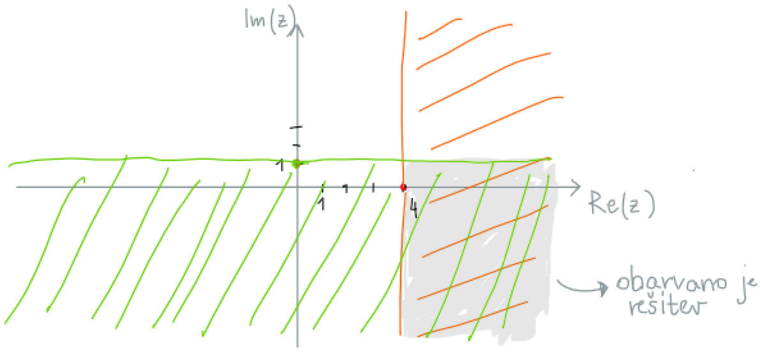
3. V kompleksni ravnini nariši množico kompleksnih števil z, ki ustrezajo pogoju

(a) $Re(z) = 3$	(e) $\lvert z\rvert \leq 2 \wedge Im(z) > -1$	(i) $\lvert z + 2i\rvert \geq 2$
(b) $Re(z) \geq 4 \wedge Im(z) \leq 1$	(f) $2 \leq \lvert z\rvert < 4 \wedge Im(z) \leq 1$	(j) $z\bar{z} = 4$
(c) $-1 \leq Re(z) < 2$	(g) $\lvert z - 2\rvert < 4$	(k) $\lvert z\rvert + z = 2 + i$
(d) $Re(z) + Im(z) = 0$	(h) $\lvert z + 2 - 3i\rvert > 2$	(l) $\lvert z - i\rvert \leq 1 \wedge \lvert z - 1\rvert \leq 1$

3. b) $Re(z) \geq 4 \wedge Im(z) \leq 1 \rightsquigarrow x \geq 4 \wedge y \leq 1$

$Re(z)$ x

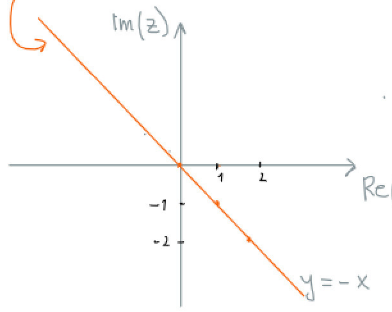
$Im(z)$.... y



d) $Re(z) + Im(z) = 0$ $\leadsto$ $x + y = 0$

$y = -x$ premica

x	y
0	0
1	-1
2	-2

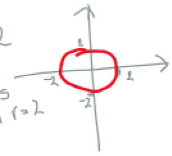


množica rešitev je samo premica

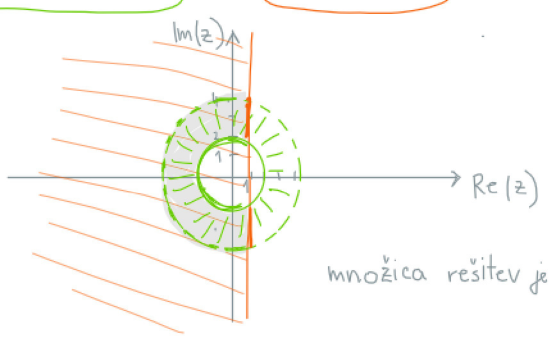
f) $\underbrace{2 \le |z| < 4}$ $\wedge$ $\underbrace{Im(z) \le 1}$

$|z|$.... oddaljenost od izhodišča (polmer krožnice)

npr. $|z| = 2$ (r)



krožnica s polmerom $r = 2$

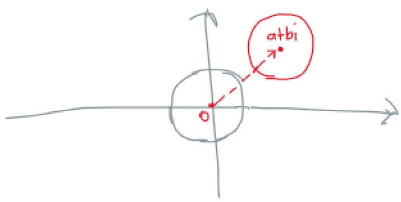


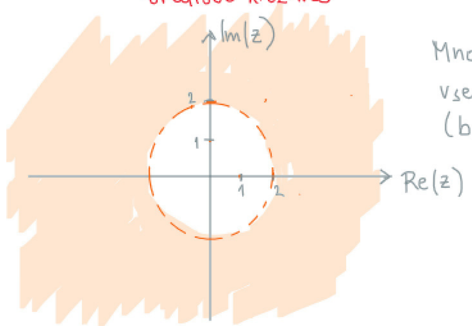
množica rešitev je obarvani lik

h) $|z + 2 - 3i| > 2$

$|z - \underbrace{(-2 + 3i)}_{\text{središče krožnice}}| > 2$ — polmer krožnice

! premik krožnice iz izhodiščne lege: $|z - (a + bi)|$





Množica rešitev je vse zunaj krožnice (brez krožnice!)

k) $|z| + z = 2 + i$ $\leadsto$ $\sqrt{x^2+y^2} + x + yi = 2 + i$

enačimo realne dele posebej in imaginarne dele posebej

$z = x + yi$

$\sqrt{x^2+y^2} + x = 2$ $\qquad$ $y\cancel{i} = \cancel{i}$ $\quad /:i$

$\sqrt{x^2+1} + x = 2$ $\qquad$ $y = 1$

vstavimo

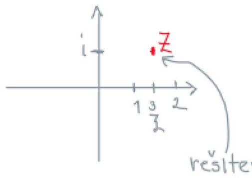
$\sqrt{x^2+1} = (2 - x) \;/^2$

$\cancel{x^2} + 1 = 4 - 2x + \cancel{x^2}$

$2x = 3$

$x = \frac{3}{2}$

Rešitev: $z = \frac{3}{2} + i$



rešitev je samo točka $\left(\frac{3}{2}, 1\right)$

4. Izračunaj.

(a) $\frac{5}{2i} + \frac{10i}{1+3i}$

(b) $\frac{2}{1-i} - \frac{3+i}{(2-i)^2}$

5. Izračunaj vrednost izraza

(a) $\frac{z-\bar{z}}{1+z\bar{z}}$, če je $z = 3+4i$

(b) $\frac{z+\bar{z}}{3z-2}$, če je $z = \frac{3-i}{3}$

6. Izračunaj kompleksno število z, za katerega je

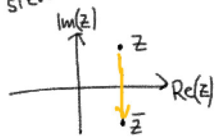
(a) $z + 2\bar{z} + i = 6$

(b) $\bar{z} + 3z - 5 = 3i(4-i)$

(c) $\bar{z} + \overline{(3-i)}(1+i^{147}) = 2(1-z) + |3-4i|$

konjugirano število z

$z = a + bi \qquad \bar{z} = a - bi$



preslikava čez x os

npr. $z = -2-4i \Rightarrow \bar{z} = -2+4i$

$|z| = \sqrt{Re(z)^2 + Im(z)^2} = \sqrt{a^2+b^2}$

npr. $z = 2-5i \Rightarrow |z| = |2-5i| = \sqrt{2^2+(-5)^2} = \sqrt{29}$

→ racionaliziramo imenovalec (znebimo se imaginarnega dela v imen.)

NAMIG:

$(a-b)(a+b) = a^2 - b^2$

ali

$(a+b)(a-b) = a^2 - b^2$

4. a) $\frac{5}{2i} + \frac{10i}{1+3i} = \frac{5\cdot i}{2i\cdot i} + \frac{10i\cdot(1-3i)}{(1+3i)\cdot(1-3i)}$

$= \frac{5i}{\underbrace{2i^2}_{2\cdot(-1)=-2}} + \frac{10i(1-3i)}{\underbrace{1^2-(3i)^2}_{1-9i^2=1+9=10}}$

$= \frac{5i}{-2} + \frac{\cancel{10}i(1-3i)}{\cancel{10}}$

$= -\frac{5i}{2} + i - 3i^2 = \underbrace{-\frac{5}{2}i + i}_{\text{združimo}} + 3 = 3 - \frac{3}{2}i$

5. b) $\frac{z+\bar{z}}{3z-2} = \quad ; \text{ kjer je } z = \frac{3-i}{3}$

$z + \bar{z} = a+bi+a-bi = 2a$

$= \frac{\overbrace{\frac{3-i}{3} + \frac{3+i}{3}}^{=2}}{3z-2} = \frac{2}{3\cdot\frac{3-i}{3}-2} = \frac{2}{\frac{9-3i}{3}-\frac{4}{2}} =$

$= \frac{2}{\frac{9-3i-4}{2}} = \frac{\frac{2}{1}}{\frac{5-3i}{2}} = \frac{4}{5-3i} =$

$= \frac{4\cdot(5+3i)}{(5-3i)\cdot(5+3i)} = \frac{4\cdot(5+3i)}{\underbrace{25-9i^2}_{25+9=34}} = \frac{\cancel{4}^{2}(5+3i)}{\cancel{34}_{17}} = \frac{2\cdot(5+3i)}{17} = \frac{10+6i}{17} = \frac{10}{17} + \frac{6}{17}i$

POSTOPEK:

① poenostavimo z: $z = \frac{3}{3} - \frac{i}{3} = 1 - \frac{1}{3}i$ ($a = 1$, $b = -\frac{1}{3}$)

② zapišemo konjugirano z: $\bar{z} = 1 + \frac{1}{3}i = \frac{3+i}{3}$

③ vstavimo z in $\bar{z}$ v izraz ter poenostavimo

6. b) $\bar{z} + 3z - 5 = 3i(4-i) \qquad z = ?$

$\left.\begin{array}{l} z = a+bi \\ \bar{z} = a-bi \end{array}\right\}$ to vstavimo v enačbo

$a - bi + 3(a+bi) - 5 = 12i - 3i^2$ ⤳ uredimo enačbo

$a - bi + 3a + 3bi = 12i + 3 + 5$ ⤳ znane vrednosti na drugo stran enačbe (ni obvezno)

$4a + 2bi = 8 + 12i$ ⤳ enačimo realne dele posebej (členi brez „i") in imaginarne dele posebej (členi z „i")

$4a = 8 \qquad 2b = 12$

$a = 2 \qquad b = 6$

REŠITEV: $z = 2 + 6i$

7. Za katera kompleksna števila z, je

(a) $z^2 + 4z$ realno število

(b) $(z - 3i)\overline{(1 - 2i)}$ imaginarno število

8. Kompleksna števila izrazi v polarni obliki in izračunaj njun produkt.

(a) $z = 1 + i\sqrt{3}$

(b) $w = 4 - 4i$

9. Izračunaj.

(a) $(\sqrt{3} - i)^{12}$

(b) $(-1 + i)^{100}$

(c) $(-3 - i\sqrt{3})^{2016}$

(d) $(2 - 2\sqrt{3}i)^{111}(3 + 3i)^{212}$

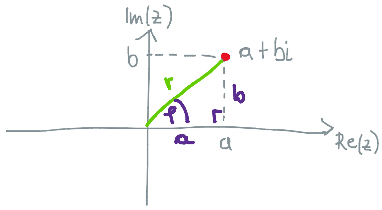
$z = a + bi$

Število je realno, ko je $\mathrm{Im}(z) = 0$ — $b = 0$

Število je imaginarno, ko je $\mathrm{Re}(z) = 0$ — $a = 0$

POLARNI ZAPIS:

$z = r\cdot(\cos\varphi + i\cdot\sin\varphi)$



$a = r\cdot\cos\varphi$

$b = r\cdot\sin\varphi$

$|z| = r = \sqrt{a^2+b^2}$

$\varphi = \mathrm{arctg}\left(\frac{b}{a}\right)$

formule dobimo iz kotnih funkcij in Pitagorovega izreka (glej skico)

7. a) $z^2 + 4z \in \mathbb{R}$ — $z = a + bi$ vstavimo

$(a+bi)^2 + 4(a+bi) =$

$= a^2 + 2abi + b^2\underbrace{i^2}_{-1} + 4a + 4bi =$ ⤳ moramo določiti imaginarni del tega števila

$= a^2 - b^2 + 4a + 2abi + 4bi =$

$= a^2 - b^2 + 4a + \underbrace{(2ab + 4b)}_{\text{imaginarni del} = 0\,!}i$

$\longrightarrow 2ab + 4b = 0$

$\underbrace{2b}_{=0}\cdot\underbrace{(a+2)}_{=0} = 0$

$2b = 0 \qquad a + 2 = 0$

$b = 0 \qquad a = -2$

$z = -2 + 0\cdot i$

REŠITEV: $z = -2$

8. b) $w = \underbrace{4}_{a} - \underbrace{4}_{b}i$

$r = \sqrt{a^2+b^2} = \sqrt{4^2+(-4)^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$

$\varphi = \mathrm{arctg}\left(\frac{b}{a}\right) = \mathrm{arctg}\left(\frac{-4}{4}\right) = \mathrm{arctg}(-1) = -\frac{\pi}{4} = -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$

REŠITEV: $w = 4\sqrt{2}\left(\cos\frac{7\pi}{4} + i\cdot\sin\frac{7\pi}{4}\right)$

9. a) $(\sqrt{3} - i)^{12}$ — razcepimo na $2\cdot 6$

$= \left((\sqrt{3} - i)^2\right)^6 = (2 - 2\sqrt{3}i)^6$ — najprej to izračunaj — dalje razcepi

$(\sqrt{3} - i)^2 = 3 - 2\sqrt{3}i + i^2$

$= 3 - 2\sqrt{3}i - 1$

$= 2 - 2\sqrt{3}i$

$= \left((2 - 2\sqrt{3}i)^2\right)^3 = (-8 - 8\sqrt{3}i)^3$

$(2 - 2\sqrt{3}i)^2 = 4 - 8\sqrt{3}i + 12i^2$

$= -8 - 8\sqrt{3}i$

$= \left(-8(1 + \sqrt{3}i)\right)^3 = (-8)^3\cdot(1 + i\sqrt{3})^3$

$= -512\cdot(1 + 3i\sqrt{3} + 3i^2\sqrt{3}^2 + i^3\sqrt{3}^3)$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$= -512\cdot(1 + \cancel{3\sqrt{3}i} - 9 - \cancel{3\sqrt{3}i})$

$= -512\cdot(-8) = 4098$

10. Reši kompleksne enačbe.

(a) $z^2 = 1 + \sqrt{3}i$

(b) $\overline{z} = z^3$

(c) $z^3 = i\sqrt{27}$

(d) $z^3 + 1 - i = 0$

(e) $z^8 - z^4 - 6 = 0$

$z = r \cdot (\cos\varphi + i \cdot \sin\varphi) \,/^n$

$z^n = r^n(\cos n\varphi + i \cdot \sin n\varphi)$

10. a) $z^2 = 1 + \sqrt{3}i \qquad z = ?$

1. način: $(a+bi)^2 = 1+\sqrt{3}i$

$a^2 + 2abi + b^2i^2 = 1+\sqrt{3}i$

$a^2 + 2abi - b^2 = 1+\sqrt{3}i$

$a^2 - b^2 + 2abi = 1+\sqrt{3}i$

$a^2 - b^2 = 1 \qquad 2ab = \sqrt{3}$

$\left(\frac{\sqrt{3}}{2b}\right)^2 - b^2 = 1 \qquad a = \frac{\sqrt{3}}{2b}$

$\vdots$

Lahko rešujemo tako, ampak je bolj zamuden način, boljše je s pomočjo polarnega zapisa

Boljši!

2. način: $\overbrace{r^2(\cos 2\varphi + i \cdot \sin 2\varphi)}^{z^2} = 1+\sqrt{3}i$

spremeni v polarni zapis

$r = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{1+3} = \sqrt{4} = 2$

$r^2(\cos 2\varphi + i \cdot \sin 2\varphi) = 2\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)$

enačimo polmera in nato še kota:

$\varphi = \operatorname{arctg}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$r^2 = 2 \qquad 2\varphi = \frac{\pi}{3}$

$r = \sqrt{2} \qquad \varphi = \frac{\pi}{6}$

$z = \sqrt{2} \cdot \left(\cos\frac{\pi}{6} + i \cdot \sin\frac{\pi}{6}\right)$ rešitev v polarnem zapisu

$z = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$ rešitev v kartezičnem zapisu

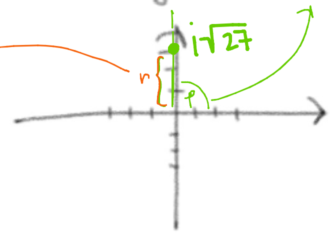
$a = r \cdot \cos\varphi \qquad b = r \cdot \sin\varphi$

$a = \sqrt{2} \cdot \cos\frac{\pi}{6} \qquad b = \sqrt{2} \cdot \sin\frac{\pi}{6}$

$a = \frac{\sqrt{6}}{2} \qquad b = \frac{\sqrt{2}}{2}$

c) $z^3 = i\sqrt{27}$

$i\sqrt{27} \Rightarrow \quad r = \sqrt{a^2+b^2} = \sqrt{0 + \sqrt{27}^2} = \sqrt{27} = 3\sqrt{3} \qquad \varphi = \operatorname{arctg}\left(\frac{\sqrt{27}}{0}\right) = \frac{\pi}{2}$



$r^3(\cos 3\varphi + i \cdot \sin 3\varphi) = 3\sqrt{3} \cdot \left(\cos\frac{\pi}{2} + i \cdot \sin\frac{\pi}{2}\right)$

enačimo: $r^3 = 3\sqrt{3} \qquad 3\varphi = \frac{\pi}{2}$

$r = \sqrt[3]{3\sqrt{3}} \qquad \varphi = \frac{\pi}{6}$

$r = \sqrt[6]{27}$

(glej SŠ - koreni višjih stopenj)

$z = \sqrt[6]{27}\left(\cos\frac{\pi}{6} + i \cdot \sin\frac{\pi}{6}\right)$ rešitev v polarnem zapisu

$z = \frac{\sqrt{3} \cdot \sqrt[6]{27}}{2} + \frac{\sqrt[6]{27}}{2}i$ rešitev v kartezičnem zapisu